Quadratic BSDEs with random terminal time and elliptic PDEs in infinite dimension.

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Abstract

In this paper we study one dimensional backward stochastic differential equations (BSDEs) with random terminal time not necessarily bounded or finite when the generator $F(t, Y, Z)$ has a quadratic growth in Z. We provide existence and uniqueness of a bounded solution of such BSDEs and, in the case of infinite horizon, regular dependence on parameters. The obtained results are then applied to prove existence and uniqueness of a mild solution to elliptic partial differential equations in Hilbert spaces. Finally we show an application to a control problem.

1 Introduction

Let τ be a stopping time which is not necessarily bounded or finite. We look for a pair of processes $(Y_t, Z_t)_{t\geq 0}$ progressively measurable which satisfy $\forall t \geq 0, \forall T \geq t$

$$\begin{cases} Y_{t\wedge\tau} = Y_{T\wedge\tau} + \displaystyle\int_{t\wedge\tau}^{T\wedge\tau} F(s, Y_s, Z_s) - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s \\ \\ Y_\tau = \xi \text{ on } \{\tau < \infty\} \end{cases} \tag{1}$$

where W is a cylindrical Wiener process in some infinite dimensional Hilbert space Ξ and the generator F has quadratic growth with respect to the variable z. Moreover the terminal condition ξ is $\mathcal{F}_\tau$-measurable and bounded. We limit ourselves to the case in which $(Y_t)_{t\geq 0}$ is one-dimensional and we look for a solution $(Y_t, Z_t)_{t\geq 0}$ such that $(Y_t)_{t\geq 0}$ is a bounded process and $(Z_t)_{t\geq 0}$ is a process with values in the space of the Hilbert-Schmidt operator from Ξ to $\mathbf{R}$ such that $\mathbb{E}\left(\int_0^{t\wedge\tau} |Z_s|^2 ds\right) < \infty, \forall t \geq 0$.

BSDEs with random terminal time have been treated by several authors (see for instance [21], [6], [3], [23]) when the generator is Lipschitz, or monotone and with suitable growth with respect to y, but Lipschitz with respect to z. Kobylanski [18] deals with a real BSDE with quadratic

generator with respect to z and with random terminal time. She requires that the stopping time is bounded or $\mathbb{P}$-a.s finite. We generalize in a certain sense the result of Kobylanski, but, to obtain the existence and uniqueness of the solution to (1) for a general stopping time, we have to require stronger assumption on the generator. In particular it has to be strictly monotone with respect to y.

We follow the techniques introduced by Briand and Hu in [3], and used successively by Royer [23], based upon an approximation procedure and on Girsanov transform. We can use this strategy even if, under our assumptions, the generator is not Lipschitz with respect to z. The main idea is to exploit the theory of BMO-martingales. It is indeed known that if (Y, Z) solves a quadratic BSDE with bounded (or $\mathbb{P}$-a.s.) finite final time then $\int_0^\cdot Z_s\, dW_s$ is a BMO–martingale (see [16]).

Then the result on BSDE is exploited to study existence and uniqueness of a mild solution (see Section 5 for the definition) to the following elliptic partial differential equation in Hilbert space H

$$\mathcal{L}u(x) + F(x, u(x), \nabla u(x)\sigma) = 0, \quad x \in H, \tag{2}$$

where F is a function from $H \times \mathbb{R} \times \Xi^*$ to $\mathbb{R}$ strictly monotone with respect the second variable and with quadratic growth in the gradient of the solution and $\mathcal{L}$ is the second order operator:

$$\mathcal{L}\phi(x) = \frac{1}{2} Trace(\sigma\sigma^* \nabla^2 \phi(x)) + \langle Ax, \nabla\phi(x)\rangle + \langle b(x), \nabla\phi(x)\rangle.$$

H is an Hilbert space, A is the generator of a strongly continuous semigroup of bounded linear operators $(e^{tA})_{t\geq 0}$ in H, b is a function with values in H and σ belongs to $L(\Xi, H)$- the space of linear bounded operator from Ξ to K satisfying appropriate Lipschitz conditions.

Existence and uniqueness of a mild solution of equation (2) in infinite dimensional spaces have been recently studied by several authors employing different techniques (see [5], [14], [9] and [10]).

In [13] (following several papers dealing with finite dimensional situations, see, for instance [4], [6] and [20]) the solution of equation (2) is represented using a Markovian forward-backward system of equations

$$\begin{cases} dX_s = AX_s ds + b(X_s)ds + \sigma(X_s)dW_s, & s \geq 0 \\ dY_s = -F(X_s, Y_s, Z_s)ds + Z_s dW_s, & s \geq 0 \\ X_0 = x \end{cases} \tag{3}$$

where F is Lipschitz with respect to y and z and monotone in y, but with monotonicity constant large. A such limitation has then been removed under certain conditions in [17], still assuming F Lipschitz with respect to z, strictly monotone and with arbitrary growth with respect to y. We follow the same approach to deal with mild solution to (2) when the coefficient F is strictly monotone in the second variable (there are not conditions on its monotonicity constant) and has quadratic growth in the gradient of the solution. The main technical point here will be proving differentiability of the bounded solution of the backward equation in system (3) with respect to the initial datum x of the forward equation. To obtain this result we follow [17]. The proof is based on an a-priori bound for suitable approximations of the equations for the gradient of Y with respect to x. We use again classical result on BMO-martingales.

In the last part of the paper we apply the above result to an optimal control problem with state equation:

$$
\begin{cases} dX_\tau = AX_\tau d\tau + b(X_\tau)d\tau + \sigma r(X_\tau, u_\tau)d\tau + \sigma dW_\tau, \\ X_0 = x \in H, \end{cases} \tag{4}
$$

where u denotes the control process, taking values in a given closed subset $\mathcal{U}$ of a Banach space U. The control problem consists of minimizing an infinite horizon cost functional of the form

$$
J(x, u) = \mathbb{E} \int_0^\infty e^{-\lambda\sigma} g(X_\sigma^u, u_\sigma) d\sigma.
$$

We suppose that r is a function with values in Ξ^* with linear growth in u and g is a given real function with quadratic growth in u. λ is any positive number. We assume that neither $\mathcal{U}$ nor r is bounded: in this way the Hamiltonian corresponding to the control problem has quadratic growth in the gradient of the solution and consequently the associated BSDE has quadratic growth in the variable Z. The results obtained on equation (2) allows to prove that the value function of the above problem is the unique mild solution of the corresponding Hamilton-Jacobi-Bellman equation (that has the same structure as (2). Moreover the optimal control is expressed in terms of a feedback that involves the gradient of that same solution to the Hamilton-Jacobi-Bellman equation. We stress that the usual application of the Girsanov technique is not allowed (since the Novikov condition is not guaranteed) and we have to use specific arguments both to prove the fundamental relation and to solve the closed loop equation. We adapt some procedure used in [11] to our infinite dimensional framework on infinite horizon.

The paper is organized as follows: the next Section is devoted to notations; in Section 3 we deal with quadratic BSDEs with random terminal time; in Section 4 we study the forward backward system on infinite horizon; in Section 5 we show the result about the solution to PDE. The last Section is devoted to the application to the control problem.

2 Notations

The norm of an element x of a Banach space E will be denoted $|x|_E$ or simply $|x|$, if no confusion is possible. If F is another Banach space, $L(E, F)$ denotes the space of bounded linear operators from E to F, endowed with the usual operator norm.

The letters Ξ, H, U will always denote Hilbert spaces. Scalar product is denoted $\langle \cdot, \cdot \rangle$, with a subscript to specify the space, if necessary. All Hilbert spaces are assumed to be real and separable. $L_2(\Xi, U)$ is the space of Hilbert-Schmidt operators from Ξ to U, endowed with the Hilbert-Schmidt norm, that makes it a separable Hilbert space. We observe that if $U = \mathbf{R}$ the space $L_2(\Xi, \mathbf{R})$ is the space $L(\Xi, \mathbf{R})$ of bounded linear operators from Ξ to $\mathbf{R}$. By the Riesz isometry the dual space $\Xi^* = L(\Xi, \mathbf{R})$ can be identified with Ξ.

By a cylindrical Wiener process with values in a Hilbert space Ξ, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we mean a family $\{W_t,\ t \geq 0\}$ of linear mappings from Ξ to $L^2(\Omega)$, denoted $\xi \mapsto \langle \xi, W_t \rangle$, such that

(i) for every $\xi \in \Xi$, $\{\langle \xi, W_t \rangle,\ t \geq 0\}$ is a real (continuous) Wiener process;

(ii) for every $\xi_1, \xi_2 \in \Xi$ and $t \geq 0$, $\mathbb{E}\left(\langle \xi_1, W_t \rangle \cdot \langle \xi_2, W_t \rangle\right) = \langle \xi_1, \xi_2 \rangle_\Xi\, t$.

$(\mathcal{F}_t)_{t \geq 0}$ will denote, the natural filtration of W, augmented with the family of $\mathbb{P}$-null sets. The filtration $(\mathcal{F}_t)$ satisfies the usual conditions. All the concepts of measurably for stochastic

processes refer to this filtration. By $\mathcal{B}(\Lambda)$ we mean the Borel σ-algebra of any topological space Λ.

We also recall notations and basic facts on a class of differentiable maps acting among Banach spaces, particularly suitable for our purposes (we refer the reader to [12] for details and properties). We notice that the use of Gâteaux differentiability in place of Fréchet differentiability is particularly suitable when dealing with evaluation (Nemitskii) type mappings on spaces of summable functions.

Let now X, Z, V denote Banach spaces. We say that a mapping $F : X \to V$ belongs to the class $\mathcal{G}^1(X, V)$ if it is continuous, Gâteaux differentiable on X, and its Gâteaux derivative $\nabla F : X \to L(X, V)$ is strongly continuous.

The last requirement is equivalent to the fact that for every $h \in X$ the map $\nabla F(\cdot)h : X \to V$ is continuous. Note that $\nabla F : X \to L(X, V)$ is not continuous in general if $L(X, V)$ is endowed with the norm operator topology; clearly, if this happens then F is Fréchet differentiable on X. It can be proved that if $F \in \mathcal{G}^1(X, V)$ then $(x, h) \mapsto \nabla F(x)h$ is continuous from $X \times X$ to V; if, in addition, G is in $\mathcal{G}^1(V, Z)$ then $G(F)$ belongs to $\mathcal{G}^1(X, Z)$ and the chain rule holds: $\nabla(G(F))(x) = \nabla G(F(x))\nabla F(x)$.

When F depends on additional arguments, the previous definitions and properties have obvious generalizations.

3 Quadratic BSDEs with random terminal time

Let τ be an $\mathcal{F}_t$-stopping time. It is not necessarily bounded or $\mathbb{P}$-a.s. finite. We work with a function F defined on $\Omega \times [0, \infty) \times \mathbf{R} \times \Xi^*$ which takes its values in $\mathbf{R}$ and such that $F(\cdot, y, z)$ is a progressively measurable process for each (y, z) in $\mathbf{R} \times \Xi^*$. We define the following sets of $\mathcal{F}_t$-progressively measurable processes $(\psi_t)_{t\geq 0}$ with values in a Hilbert space K:

$$\mathcal{M}^{2,-2\lambda}(0, \tau; K) = \left\{ \psi : \mathbb{E}\left(\int_0^\tau e^{-2\lambda s}|\psi_s|^2 ds \right) < \infty \right\},$$

$$\mathcal{M}^2_{loc}(0, \tau; K) = \left\{ \psi : \mathbb{E}\left(\int_0^{t\wedge\tau} |\psi_s|^2 ds \right) < \infty \quad \forall t \geq 0 \right\}.$$

We want to construct an adapted process $(Y, Z)_{t\geq 0}$ which solves the BSDE

$$-\,dY_t = \mathbf{1}_{t\leq\tau}(F(t, Y_t, Z_t)dt - Z_t dW_t), \quad Y_\tau = \xi \text{ on } \{\tau < \infty\}. \tag{5}$$

We assume that:

Assumption A1. There exist $C \geq 0$ and $\alpha \in (0, 1)$ such that

1. $|F(t, y, z)| \leq C\left(1 + |y| + |z|^2\right)$;
2. $F(t, \cdot, \cdot)$ is $\mathcal{G}^{1,1}(\mathbf{R} \times L_2(\Xi, \mathbf{R}); \mathbf{R})$;
3. $|\nabla_z F(t, y, z)| \leq C\,(1 + |z|)$;
4. $|\nabla_y F(t, y, z)| \leq C\,(1 + |z|)^{2\alpha}$.

   Moreover we suppose that there exist two constants $K \geq 0$ and $\lambda > 0$ such that $d\mathbb{P} \otimes dt$ a.e.:

5. F is monotone in y in the following sense:

$$\forall y, y' \in \mathbf{R}, z \in \Xi^*, \quad < y - y', F(t, y, z) - F(t, y', z) > \leq -\lambda |y - y'|^2;$$

6. $|F(t, 0, 0)| \leq K$;

7. ξ is a $\mathcal{F}_\tau$-measurable bounded random variable; we denote by M some real such that $|\xi| \leq M$ $\mathbb{P}$-a.s.

We call solution of the equation a pair of progressively measurable processes $(Y_t, Z_t)_{t \geq 0}$ with values in $\mathbf{R} \times \Xi^*$ such that

1. Y is a bounded process and $Z \in \mathcal{M}^2_{loc}(0, \tau; \Xi^*)$;

2. On the set $\{\tau < \infty\}$, we have $Y_\tau = \xi$ and $Z_t = 0$ for $t > \tau$;

3. $\forall T \geq 0$, $\forall t \in [0, T]$ we have $Y_{t \wedge \tau} = Y_{T \wedge \tau} + \int_{t \wedge \tau}^{T \wedge \tau} F(s, Y_s, Z_s) ds - \int_{t \wedge \tau}^{T \wedge \tau} Z_s dW_s$.

Before giving the main result of this section we prove a lemma which we use in the sequel. The proof involves the Girsanov transform and results of the bounded mean oscillation (BMO, for short) martingales theory.

Here we recall a few well-known facts from this theory following the exposition in [15]. Let M be a continuous local $(P, \mathcal{F})$-martingale satisfying $M_0 = 0$. Let $1 \leq p < \infty$. Then M is in the normed linear space BMOp if

$$||M||_{BMO_p} = \sup_\tau \left|\left| \mathbb{E}[|M_T - M_\tau|^p | \mathcal{F}_\tau]^{1/p} \right|\right|_\infty < \infty,$$

where the supremum is taken over all stopping time $\tau \leq T$. By Corollary 2.1 in [15], M is a BMOp-martingale if and only if it is a BMOq-martingale for every $q \geq 1$. Therefore, it is simply called a BMO-martingale. In particular, M is a BMO-martingale if and only if

$$||M||_{BMO_2} = \sup_\tau \left|\left| \mathbb{E}[\langle M \rangle_T - \langle M \rangle_\tau | \mathcal{F}_\tau]^{1/2} \right|\right|_\infty < \infty,$$

where the supremum is taken over all stopping time $\tau \leq T$; $\langle M \rangle$ denotes the quadratic variation of M. This means that local martingales of the form $M_t = \int_0^t \xi_s dW_s$ are BMO-martingales if and only if

$$||M||_{BMO_2} = \sup_\tau \left|\left| \mathbb{E}\left[\int_\tau^T ||\xi_s||^2 ds \Big| \mathcal{F}_\tau \right]^{1/2} \right|\right|_\infty < \infty.$$

The very important feature of BMO-martingales is the following (see Theorem 2.3 in [15]): the exponential martingale

$$\mathcal{E}(M)_t = \mathcal{E}_t = \exp\left(M_t - \frac{1}{2} \langle M \rangle_t \right) \quad 0 \leq t \geq T$$

is a uniformly integrable martingale.

Lemma 3.1. *Let (U, V), be solutions to*

$$U_t = \xi + \int_t^T \mathbf{1}_{s \leq \tau} [a_s U_s + b_s V_s + \psi_s] ds - \int_t^T V_s \, dW_s \tag{6}$$

where ξ is $\mathcal{F}_\tau$–measurable and bounded and a_s, b_s, ψ_s are processes such that

1) $a_s \leq -\lambda$ *for some* $\lambda > 0$*;*

2) $\int_0^{\cdot} b_s dW_s$ *is a BMO-martingale;*

3) $|\psi_s| \leq \rho(s)$ *where* ρ *is a deterministic function.*

Moreover we assume that U *is bounded. Then we have* $\mathbb{P}$*-a.s. for all* $t \in [0, T]$

$$|U_t| \leq e^{-\lambda(T-t)} \|\xi\|_\infty + \int_t^T \rho(s) e^{-\lambda(s-t)}\, ds.$$

Proof. Let (U, V) be a solution of the BSDE (6) such that U is bounded.

We fix $t \in \mathbf{R}_+$ and set for $s \geq t$ $e_s = e^{\int_{t\wedge\tau}^{s\wedge\tau} a_r\, dr}$. By Ito's formula we have,

$$U_t = e_T \xi + \int_t^T \mathbf{1}_{s\leq\tau} e_s \psi_s ds - \int_t^T e_s V_s (dW_s - b_s).$$

Let $\mathbb{Q}_T$ the probability measure on $(\Omega, \mathcal{F}_T)$ whose density with respect to $\mathbb{P}_{|\mathcal{F}_T}$ is

$$\mathcal{E}_T = \exp\left(\int_0^T b_s dW_s - \frac{1}{2} \int_0^T |b_s|^2 ds \right).$$

By assumption $\int_0^{\cdot} b_s dW_s$ is a BMO-martingale and the probability measures $\mathbb{Q}_T$ and $\mathbb{P}_{|\mathcal{F}_T}$ are mutually absolutely continuous and $\overline{W}_t = W_t - \int_0^t b_r\, dr$ for $0 \leq t \leq T$ is a Brownian motion under $\mathbb{Q}_T$.

Taking the conditional expectation with respect to $\mathcal{F}_t$ we get

$$|U_t| \leq \mathbb{E}^{\mathbb{Q}_T} \left[e_T |\xi| + \int_t^T e_s |\psi_s| ds \,\Big|\, \mathcal{F}_t \right], \quad \mathbb{Q}_T \text{ a.s.}$$

and thanks to 3)

$$|U_t| \leq (\mathcal{E}_t)^{-1} \mathbb{E} \left(\mathcal{E}_T e_T |\xi| + \int_t^T \rho(s) e_s ds \,\Big|\, \mathcal{F}_t \right).$$

But from 1) $a_s \leq -\lambda$ and, for all $s \geq t$ $e_s \leq e^{-\lambda(s-t)}$ $\mathbb{P}$-a.s., from which we get $\mathbb{P}$-a.s. $\forall t \in [0, T]$

$$|U_t| \leq e^{-\lambda(T-t)} ||\xi||_\infty + \int_t^T \rho(s) e^{-\lambda(s-t)} ds.$$

$\square$

Corollary 3.2. *Let* (Y^i, Z^i)*,* $i = 1, 2$*, be solutions to*

$$Y_t^i = \xi^i + \int_t^T \mathbf{1}_{s\leq\tau} F^i(s, Y_s^i, Z_s^i)\, ds - \int_t^T Z_s^i\, dW_s$$

where ξ^i *is* $\mathcal{F}_\tau$*–measurable and bounded. We assume that* Y^1 *and* Y^2 *are bounded and that the* Z^i *are such that* $\int_0^{\cdot} Z_d^i dW_t$ *are BMO-martingales. Moreover* F^1 *is* $-\lambda$*-monotone in the following sense: there exists* $\lambda > 0$ *such that*

$$\forall y, y' \in \mathbf{R}, z \in \Xi^*, \quad < y - y', F^1(t, y, z) - F^1(t, y', z) > \leq -\lambda |y - y'|^2;$$

and verifies

$$|F^1(t,y,z) - F^1(t,y,z')| \leq C\,|z-z'|\left(1+|z|+|z'|\right).$$

We assume moreover that

$$|F^1(t,Y_t^2,Z_t^2) - F^2(t,Y_t^2,Z_t^2)| \leq \rho(t)$$

where ρ is a deterministic function. Then we have $\mathbb{P}$-a.s. for all $t \in [0,T]$

$$|Y_t^1 - Y_t^2| \leq e^{-\lambda(T-t)}\|\xi^1-\xi^2\|_\infty + \int_t^T \rho(s)e^{-\lambda(s-t)}\,ds.$$

Proof. Let (Y^1,Z^1) and (Y^2,Z^2) be solutions of the BSDE with data respectively (ξ^1,F^1) and (ξ^2,F^2) such that Y^1 and Y^2 are bounded. We set $\overline{Y} = Y^1 - Y^2$ and $\overline{Z} = Z^1 - Z^2$. It is enough to write the equation for the difference $\overline{Y} = Y^1 - Y^2$

$$d\overline{Y}_t = -\mathbf{1}_{t\leq\tau}[F^1(t,Y_t^1,Z_t^1) - F^2(t,Y_t^2,Z_t^2)dt + \overline{Z}_t dW_t]$$

as

$$d\overline{Y}_t = -\mathbf{1}_{t\leq\tau}[(a_t\overline{Y}_t + b_t\overline{Z}_t + \psi_t)dt + \overline{Z}_t dW_t].$$

using a linearization procedure by setting

$$a_s = \begin{cases} \dfrac{F^1(s,Y_s^1,Z_s^1) - F^1(s,Y_s^2,Z_s^1)}{Y_s^1 - Y_s^2}, & \text{if } Y_s^1 - Y_s^2 \neq 0 \\ -\lambda & \text{otherwise} \end{cases}$$

$$b_s = \begin{cases} \dfrac{F^1(s,Y_s^2,Z_s^1) - F^1(s,Y_s^2,Z_s^2)}{|Z_s^1 - Z_s^2|^2}(Z_s^1 - Z_s^2), & \text{if } Z_s^1 - Z_s^2 \neq 0 \\ 0 & \text{otherwise .} \end{cases}$$

and

$$\psi_s = F^1(s,Y_s^2,Z_s^2) - F^2(s,Y_s^2,Z_s^2)$$

□

Now we can state the main result of this section, concerning the existence and uniqueness of solutions of BSDE (5).

Theorem 3.3. *Under assumption A1 there exists a unique solution (Y,Z) to BSDE (5) such that Y is a continuous and bounded process and Z belongs to $\mathcal{M}^2_{loc}(0,\tau;\Xi^*)$.*

Proof. **Existence.** We adopt the same strategy as in [3] and [23], with some significant modifications.

Denote by (Y^n,Z^n) the unique solution to the BSDE

$$Y_t^n = \xi\mathbf{1}_{\tau\leq n} + \int_t^n \mathbf{1}_{s\leq\tau}F(s,Y_s^n,Z_s^n)ds - \int_t^n Z_s^n dW_s, \qquad 0\leq t\leq n. \tag{7}$$

We know from results of [18] that under A1-1,2,3,4 the BSDE (7) has a unique bounded solution and that

$$\left\|\sup_{t\in[0,\tau\wedge n]}|Y_t|^n\right\|_\infty \leq (||\xi||_\infty + Cn)e^{Cn}$$

and there exists a constant $C = C_n$, which depends on $\left\|\sup_{t\in[0,\tau\wedge n]} |Y_t^n|\right\|_\infty$, such that

$$\left\|\int_0^\cdot Z_s^n \cdot dW_s\right\|_{BMO_2} \leq C_n.$$

Now we study the convergence of the sequence of processes (Y^n, Z^n).

(i) First of all we prove that, thanks to the assumptions of boundedness and monotonicity A1-5,6, Y^n is a process bounded by a constant independent on n. Applying the Corollary 3.2 we have that $\mathbb{P}$-a.s. $\forall n \in \mathbf{N}$, $\forall t \in [0,n]$

$$|Y_t^n| \leq e^{-\lambda(n-t)}||\xi \mathbf{1}_{\tau\leq n}||_\infty + \int_t^n e^{-\lambda(s-t)}|F(s,0,0)|\,ds \leq M + \frac{K}{\lambda}. \tag{8}$$

Moreover we can show that for each $\epsilon > 0$

$$\sup_{n\geq 1} \mathbb{E}(\int_0^\tau e^{-\epsilon s}|Z_s^n|^2 ds) < \infty. \tag{9}$$

To obtain this estimate we take the function $\varphi(x) = \left(e^{2Cx} - 2Cx - 1\right)/(2C^2)$ which has the following properties:

$$\varphi'(x) \geq 0 \text{ if } x \geq 0,$$
$$\frac{1}{2}\varphi''(x) - C\varphi'(x) = 1.$$

Thanks to (8) we can say that there exist a constant K_0 such that $\forall s \in [0,T]$, $Y_s^n + K_0 \geq 0$, $\mathbb{P}$-a.s. Now, if we calculate the Ito differential of $e^{-\epsilon t}\varphi(Y_t^n + K_0)$, using the previous properties, we have (9).

(ii) Now we prove that the sequence $(Y_t^n)_{n\geq 0}$ converges almost surely. We are going to show that it is an almost definite Cauchy sequence.

We define Y^n and Z^n on the whole time axis by setting

$$Y_t^n = \xi \mathbf{1}_{\tau\leq n}, \quad Z_t^n = 0, \quad \text{ if } t > n.$$

Fix $t \leq n \leq m$ and set $\widehat{Y} = Y^m - Y^n$, $\widehat{Z} = Z^m - Z^n$ and $\widehat{F}(s,y,z) = \mathbf{1}_{s\leq n}F(s,y,z)$. We get, from Ito's formula

$$\widehat{Y}_t = \widehat{Y}_m + \int_t^m \mathbf{1}_{s\leq\tau}(F(s,Y_s^m,Z_s^m) - \widehat{F}(s,Y_s^n,Z_s^n))ds - \int_t^m \widehat{Z}_s dW_s.$$

We note that

$$|F(s,Y_s^n,Z_s^n) - \widehat{F}(s,Y_s^n,Z_s^n))| = |\mathbf{1}_{s>n}F(s,\xi\mathbf{1}_{\tau\leq n},0)| \leq C(1+M)\mathbf{1}_{s>n}.$$

Hence, we can apply the Corollary 3.2 with $\xi^1 = \xi\mathbf{1}_{\tau\leq m}$ and $\xi^2 = \xi\mathbf{1}_{\tau\leq n}$, $F^1 = F$ and $F^2 = \widehat{F}$, $\rho(t) = C(1+M)\mathbf{1}_{s>n}$ and state that $\forall n, m \in \mathbf{N}$, with $n \leq m$ and $\forall t \in [0,n]$, $\mathbb{P}$-a.s.

$$|Y_t^m - Y_t^n| \leq e^{-\lambda(m-t)}||\xi\mathbf{1}_{\tau\leq m} - \xi\mathbf{1}_{\tau\leq n}||_\infty + \int_n^m C(1+M)e^{-\lambda(s-t)}\,ds \leq$$
$$\leq \left(M + \frac{C(1+M)}{\lambda}\right)e^{-\lambda(n-t)}. \tag{10}$$

The previous inequality implies that for each $t \geq 0$ the sequence of random variable Y_t^n is a Cauchy sequence in $L^\infty(\Omega)$, hence converges to a limit, which we denote Y_t. If m goes to infinity in the last inequality, it comes that $\mathbb{P}$-a.s., $\forall\, 0 \leq t \leq n$

$$|Y_t^n - Y_t| \leq \beta e^{-\lambda(n-t)}, \qquad \text{where } \beta = M + \frac{C(1+M)}{\lambda}. \tag{11}$$

This inequality implies that the sequence of continuous processes $(Y^n)_{n\in\mathbf{N}}$ converges almost surely to Y uniformly with respect to t on compact sets. The limit process Y is also continuous and from (8) we have that $\forall t \in \mathbf{R}_+$ $|Y_t| \leq M + \frac{K}{\lambda}$.

(iii) We show that the sequence $(Y_n)_n$ also converges in the space $\mathcal{M}^{2,-2\lambda}(0,\tau;\mathbf{R})$. Indeed we have

$$\mathbb{E}\left[\int_0^\tau e^{-2\lambda t}|Y_t^n - Y_t|^2 dt\right] = \mathbb{E}\left[\int_0^{n\wedge\tau} e^{-2\lambda t}|Y_t^n - Y_t|^2 dt\right] + \mathbb{E}\left[\int_{n\wedge\tau}^\tau e^{-2\lambda t}|Y_t^n - Y_t|^2 dt\right]$$

and using the inequality (11) for the first term, we get that

$$\mathbb{E}\left[\int_0^{n\wedge\tau} e^{-2\lambda t}|Y_t^n - Y_t|^2 dt\right] \leq \beta^2 n e^{-2\lambda n}.$$

In addition, from the definition of Y_t^n on $\mathbf{R}_+$, we know that $\forall t > n$ $Y_t^n = \xi \mathbf{1}_{\tau\leq n}$. Hence,

$$\begin{aligned}\mathbb{E}\left[\int_{n\wedge\tau}^\tau e^{-2\lambda t}|Y_t^n - Y_t|^2 dt\right] &= \mathbb{E}\left[\mathbf{1}_{n<\tau}\int_n^\tau e^{-2\lambda t}|Y_t - \xi\mathbf{1}_{n<\tau}|^2 dt\right] \leq \\ &\leq 4\mathbb{E}\left[\mathbf{1}_{n<\tau}\left(M + \frac{K}{\lambda}\right)^2 \int_n^\tau e^{-2\lambda t} dt\right] \leq \frac{2}{\lambda}\left(M + \frac{K}{\lambda}\right)^2 e^{-2\lambda n}.\end{aligned}$$

Finally we have

$$\mathbb{E}\left[\int_0^\tau e^{-2\lambda t}|Y_t^n - Y_t|^2 dt\right] \leq e^{-2\lambda n}\left(n\beta^2 + \frac{2}{\lambda}\left(M + \frac{K}{\lambda}\right)^2\right).$$

Hence (Y^n) converges to Y in $\mathcal{M}^{2,-2\lambda}(0,\tau;\mathbf{R})$.

(iv) To continue, we show that the sequence $(Z_n)_n$ is a Cauchy sequence in the space $\mathcal{M}^{2,-2(\lambda+\epsilon)}(0,\tau;\Xi^*)$.

Fix $t \leq n \leq m$ and set, as before, $\widehat{Y} = Y^m - Y^n$, $\widehat{Z} = Z^m - Z^n$ and $\widehat{F}(s,y,z) = \mathbf{1}_{s\leq n}F(s,y,z)$. We write

$$F(s, Y_s^m, Z_s^m) - \widehat{F}(s, Y_s^n, Z_s^n) = a_s^{n,m}\widehat{Y}_s + b_s^{n,m}\widehat{Z}_s + \mathbf{1}_{s>n}F(s, \xi\mathbf{1}_{\tau\leq n}, 0)$$

where

$$a_s^{n,m} = \begin{cases} \dfrac{F(s,Y_s^m,Z_s^m) - F(s,Y_s^n,Z_s^m)}{Y_s^m - Y_s^n}, & \text{if } Y_s^m - Y_s^n \neq 0 \\ -\lambda & \text{otherwise} \end{cases}$$

$$b_s^{n,m} = \begin{cases} \dfrac{F(s,Y_s^n,Z_s^m) - F(s,Y_s^n,Z_s^n)}{|Z_s^m - Z_s^n|^2}(Z_s^m - Z_s^n), & \text{if } Z_s^m - Z_s^n \neq 0 \\ 0 & \text{otherwise .} \end{cases}$$

From Ito's formula we get

$$
\begin{aligned}
|\widehat{Y}_0|^2 &+ \int_0^{\tau\wedge m} e^{-2(\lambda+\epsilon)s}|\widehat{Z}_s|^2\, ds + \int_0^{\tau\wedge m} 2e^{-2(\lambda+\epsilon)s}\widehat{Y}_s\widehat{Z}_s dW_s = \\
&= e^{-2(\lambda+\epsilon)\tau\wedge m}|\widehat{Y}_{\tau\wedge m}|^2 + \int_0^{\tau\wedge m} e^{-2(\lambda+\epsilon)s}2(\lambda+\epsilon)|\widehat{Y}_s|^2 ds + \\
&+ \int_0^{\tau\wedge m} 2e^{-2(\lambda+\epsilon)s}\widehat{Y}_s[a_s^{n,m}\widehat{Y}_s + b_s^{n,m}\widehat{Z}_s]ds + \int_{\tau\wedge n}^{\tau\wedge m} 2e^{-2(\lambda+\epsilon)s}\widehat{Y}_s F(s,\xi\mathbf{1}_{\tau\leq n},0)ds
\end{aligned}
$$

and taking the expectation we have

$$
\begin{aligned}
\mathbb{E}\int_0^{\tau\wedge m} e^{-2(\lambda+\epsilon)s}|\widehat{Z}_s|^2 ds &\leq \mathbb{E}e^{-2(\lambda+\epsilon)\tau\wedge m}|\widehat{Y}_{\tau\wedge m}|^2 + \mathbb{E}\int_0^{\tau\wedge m} e^{-2(\lambda+\epsilon)s}2\epsilon|\widehat{Y}_s|^2 ds + \\
&+ \mathbb{E}\int_0^{\tau\wedge m} 2e^{-2(\lambda+\epsilon)s}\widehat{Y}_s b_s^{n,m}\widehat{Z}_s ds + \mathbb{E}\int_{\tau\wedge n}^{\tau\wedge m} 2e^{-2(\lambda+\epsilon)s}\widehat{Y}_s F(s,\xi\mathbf{1}_{\tau\leq n},0)ds.
\end{aligned}
$$

Using the fact that

$$
2e^{-2(\lambda+\epsilon)s}\widehat{Y}_s b_s^{n,m}\widehat{Z}_s \leq 2|\widehat{Y}_s|^2 e^{-2(\lambda+\epsilon)s}|b_s^{n,m}|^2 + \frac{1}{2}e^{-2(\lambda+\epsilon)s}|\widehat{Z}_s|^2
$$

we get

$$
\begin{aligned}
\mathbb{E}\int_0^{\tau\wedge m} e^{-2(\lambda+\epsilon)s}|\widehat{Z}_s|^2\, ds &\leq 2\mathbb{E}e^{-2(\lambda+\epsilon)\tau\wedge m}|\widehat{Y}_{\tau\wedge m}|^2 + 2\mathbb{E}\int_0^{\tau\wedge m} e^{-2(\lambda+\epsilon)s}2\epsilon|\widehat{Y}_s|^2 ds + \\
&+ 4\mathbb{E}\int_0^{\tau\wedge m} |\widehat{Y}_s|^2\, e^{-2(\lambda+\epsilon)s}|b_s^{n,m}|^2 ds + \mathbb{E}\int_{\tau\wedge n}^{\tau\wedge m} 4e^{-2(\lambda+\epsilon)s}|\widehat{Y}_s||F(s,\xi\mathbf{1}_{\tau\leq n},0)|ds \leq \\
&\leq M^2 e^{-2(\lambda+\epsilon)n} + \beta^2 e^{-2\lambda n}\left(1 + 4\int_0^{\tau\wedge m} e^{-2\epsilon s}|b_s^{n,m}|^2 ds + 4C(1+M)\mathbb{E}\int_0^{\tau\wedge m} e^{-2\lambda s}|\widehat{Y}_s|\right).
\end{aligned}
$$

We note that

$$
|b_s^{n,m}|^2 \leq C(1 + |Z_s^n|^2 + |Z_s^m|^2)
$$

and by (9) $\sup_{n\geq 1} E\int_0^\tau e^{-2\epsilon s}|Z_s^n|^2 ds < \infty$. Finally we obtain

$$
\mathbb{E}\int_0^{\tau\wedge m} e^{-2(\lambda+\epsilon)s}|\widehat{Z}_s|^2\, ds \leq \beta'(1+n)e^{-2\lambda n}
$$

where β' depends on M, λ, K. Moreover we have that

$$
\mathbb{E}\left(\int_{m\wedge\tau}^{\tau} e^{-2(\lambda+\epsilon)s}|\widehat{Z}_s|^2\, ds\right) = 0
$$

hence

$$
\mathbb{E}\left(\int_0^{\tau} e^{-2(\lambda+\epsilon)s}|\widehat{Z}_s|^2\, ds\right) \leq \beta'(1+n)e^{-2\lambda n}.
$$

Hence (Z^n) is a Cauchy sequence in $\mathcal{M}^{2,-2(\lambda+\epsilon)}(0,\tau;\Xi^*)$ and converges to the process Z in this space.

(v) It remains to show that the process (Y, Z) satisfies the BSDE (5).

We already know that Y is continuous and bounded and Z belongs to $\mathcal{M}^{2,-2(\lambda+\epsilon)}(0,\tau;\Xi^*)$.

By definition $\forall n \in \mathbf{N}$, $\forall T, t$ such that $0 \le t \le T \le n$ we have

$$Y^n_{t\wedge\tau} - Y^n_{T\wedge\tau} = \int_{t\wedge\tau}^{T\wedge\tau} F(s, Y^n_s, Z^n_s) - \int_{t\wedge\tau}^{T\wedge\tau} Z^n_s dW_s. \tag{12}$$

Fix t and T. We shall pass to the limit in L^1 in the previous equality. The sequence $Y^n_{t\wedge\tau}$ converges almost surely to Y_t and is bounded by $M + \frac{K}{\lambda}$ uniformly in n. From Lebesgue's theorem we get that the sequence converges to $Y_{t\wedge\tau}$ in L^1. Moreover, $\int_{t\wedge\tau}^{T\wedge\tau} Z^n_s dW_s$ converges in to $\int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s$ in L^2 since

$$\mathbb{E}\left(\int_{t\wedge\tau}^{T\wedge\tau} Z^n_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s\right)^2 \le e^{2(\lambda+\epsilon)T}\mathbb{E}\int_0^{T\wedge\tau} e^{-2(\lambda+\epsilon)s}|Z^n_s - Z_s|^2 ds.$$

We can note that $\int_{t\wedge\tau}^{T\wedge\tau} F(s, Y^n_s, Z^n_s)ds$ converges to $\int_{t\wedge\tau}^{T\wedge\tau} F(s, Y_s, Z_s)ds$ in L^1. Indeed

$$\mathbb{E}\left|\int_{t\wedge\tau}^{T\wedge\tau} F(s, Y^n_s, Z^n_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} F(s, Y_s, Z_s)ds\right| \le \mathbb{E}\int_0^T |F(s, Y^n_s, Z^n_s)ds - F(s, Y_s, Z_s)|ds$$

and, by the growth assumption on F, the map $(Y, Z) \to F(\cdot, Y, Z)$ is continuous from the space $L^1(\Omega; L^1([0,T];\mathbf{R})) \times L^2(\Omega; L^2([0,T];\Xi^*)$ to $L^1(\Omega; L^1([0,T];\mathbf{R}))$. (By classical result on continuity of evaluation operators, see e.g. [1]). Hence, passing to the limit in the equation (12), we obtain $\forall t, T$ such that $t \le T$

$$Y_{t\wedge\tau} - Y_{T\wedge\tau} = \int_{t\wedge\tau}^{T\wedge\tau} F(s, Y_s, Z_s) - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s.$$

So to conclude the proof, it only remains to check the terminal condition. Let $\omega \in \{\tau < \infty\}$, and $n \in \mathbf{N}$ such that $n \ge \tau(\omega)$. Then

$$\begin{aligned} |Y_\tau - \xi\mathbf{1}_{t\le 2n}|(\omega) &= |Y_{n\wedge\tau} - \xi\mathbf{1}_{t\le 2n}|(\omega) \le |Y_{n\wedge\tau} - Y^{2n}_{n\wedge\tau}|(\omega) + |Y^{2n}_{n\wedge\tau} - \xi\mathbf{1}_{t\le 2n}|(\omega) \le \\ &\le \beta e^{\lambda(n\wedge\tau)}(\omega)e^{-2\lambda n} + |Y^{2n}_{n\wedge\tau} - \xi\mathbf{1}_{t\le 2n}|(\omega) \le \beta e^{-\lambda n} \end{aligned}$$

since $Y^{2n}_{n\wedge\tau} = Y^{2n}_\tau = Y^{2n}_{2n} = \xi\mathbf{1}_{t\le 2n}$ Then, $Y_\tau = \xi$ $\mathbb{P}$-a.s. on the set $\{\tau < \infty\}$, and the process (Y, Z) is solution for BSDE (5).

Uniqueness.

Suppose that (Y^1, Z^1) and (Y^2, Z^2) are both solutions of the BSDE (5) such that Y^1 and Y^2 are continuous and bounded and Z^1 and Z^2 belong to $\mathcal{M}^2_{loc}(0, \tau; \Xi^*)$. It follows directly from the Corollary 3.2 that $\forall t \ge 0$

$$Y^1_t - Y^2_t = 0 \qquad \mathbb{P}\text{-a.s.}$$

and then, by continuity, $Y^1 = Y^2$.

Applying Ito's formula we have that $d\mathbb{P} \otimes dt$-a.e. $Z^1_t = Z^2_t$.

□

4 The forward-backward system on infinite horizon

In this Section we use the previous result to study a forward-backward system on infinite horizon, when the backward equation has quadratic generator.

We introduce now some classes of stochastic processes with values in a Hilbert space K which we use in the sequel.

- $L^p(\Omega; L^2(0,s;K))$ defined for $s \in]0,+\infty]$ and $p \in [1,\infty)$, denotes the space of equivalence classes of progressively measurable processes $\psi : \Omega \times [0,s[\to K$, such that

$$|\psi|^p_{L^p(\Omega;L^2(0,s;K))} = \mathbb{E}\left(\int_0^s |\psi_r|^2_K \, dr\right)^{p/2}.$$

  Elements of $L^p(\Omega; L^2(0,s;K))$ are identified up to modification.

- $L^p(\Omega; C(0,s;K))$, defined for $s \in]0,+\infty[$ and $p \in [1,\infty[$, denotes the space of progressively measurable processes $\{\psi_t,\, t \in [0,s]\}$ with continuous paths in K, such that the norm

$$|\psi|^p_{L^p(\Omega;C([0,s];K))} = \mathbb{E} \sup_{r\in[0,s]} |\psi_r|^p_K$$

  is finite. Elements of $L^p(\Omega; C(0,s;K))$ are identified up to indistinguishability.

- $L^2_{\text{loc}}(\Omega; L^2(0,\infty;K))$ denotes the space of equivalence classes of progressively measurable processes $\psi : \Omega \times [0,\infty) \to K$ such that

$$\forall t > 0 \quad \mathbb{E}\int_0^t |\psi_r|^2 dr < \infty.$$

Now we consider the Itô stochastic equation for an unknown process $\{X_s, s \geq 0\}$ with values in a Hilbert space H:

$$X_s = e^{sA}x + \int_0^s e^{(s-r)A} b(X_r)dr + \int_0^s e^{(s-r)A}\sigma dW_r, \;\; s \geq 0. \tag{13}$$

Our assumptions will be the following:

Assumption A2. (i) The operator A is the generator of a strongly continuous semigroup e^{tA}, $t \geq 0$, in a Hilbert space H. We denote by m and a two constants such that $|e^{tA}| \leq me^{at}$ for $t \geq 0$.

(ii) $b : H \to H$ satisfies, for some constant $L > 0$,

$$|b(x) - b(y)| \leq L|x-y|, \;\; x, y \in H.$$

(iii) σ belongs to $L(\Xi, H)$ such that $e^{tA}\sigma \in L_2(\Xi, H)$ for every $t > 0$, and

$$|e^{tA}\sigma|_{L_2(\Xi,H)} \leq Lt^{-\gamma}e^{at},$$

for some constants $L > 0$ and $\gamma \in [0, 1/2)$.

(iv) We have $b(\cdot) \in \mathcal{G}^1(H, H)$.

(v) Operators $A + b_x(x)$ are dissipative (that is $\langle Ay, y\rangle + \langle b_x(x)y, y\rangle \leq 0$ for all $x \in H$ and $y \in D(A)$).

Remark 4.1. We note we need of assumptions $(iv) - (v)$ to obtain a result of regularity of the process X with respect to initial condition x.

We start by recalling a well known result on solvability of equation (13) on a bounded interval, see e.g. [12].

Proposition 4.2. *Under the assumption A2, for every $p \in [2, \infty)$ and $T > 0$ there exists a unique process $X^x \in L^p(\Omega; C(0,T;H))$ solution of (13). Moreover, for all fixed $T > 0$, the map $x \to X^x$ is continuous from H to $L^p(\Omega; C(0,T;H))$.*

$$\mathbb{E} \sup_{r \in [0,T]} |X_r|^p \leq C(1 + |x|)^p,$$

for some constant C depending only on q, γ, T, L, a and m.

We need to state a regularity result on the process X. The proof of the following lemma can be found in [17].

Lemma 4.3. *Under Assumptions A2 the map $x \to X^x$ is Gâteaux differentiable (that is belongs to $\mathcal{G}(H, L^p(\Omega, C(0,T;H)))$). Moreover denoting by $\nabla_x X^x$ the partial Gâteaux derivative, then for every direction $h \in H$, the directional derivative process $\nabla_x X^x h, t \in \mathbb{R}$, solves, $\mathbb{P} - a.s.$, the equation*

$$\nabla_x X_t^x h = e^{tA} h + \int_0^t e^{\sigma A} \nabla_x F(X_\sigma^x) \nabla_x X_\sigma^x h \, d\sigma, \quad t \in \mathbb{R}^+.$$

Finally, $\mathbb{P}$-a.s., $|\nabla_x X_t^x h| \leq |h|$, for all $t > 0$.

The associated BSDE is:

$$Y_t^x = Y_T^x + \int_t^T F(X_\sigma^x, Y_\sigma^x, Z_\sigma^x) d\sigma - \int_t^T Z_\sigma^x dW_\sigma, \quad 0 \leq t \leq T < \infty. \tag{14}$$

Here X^x is the unique mild solution to (13) starting from $X_0 = x$. Y is real valued and Z takes values in Ξ^*, $F : H \times \mathbb{R} \times \Xi^* \to \mathbb{R}$ is a given measurable function.

We assume the following on F:

Assumption A3. There exist $C \geq 0$ and $\alpha \in (0,1)$ such that

1. $|F(x,y,z)| \leq C \left(1 + |y| + |z|^2\right)$;
2. $F(\cdot,\cdot,\cdot)$ is $\mathcal{G}^{1,1,1}(H \times \mathbf{R} \times \Xi^*; \mathbf{R})$;
3. $|\nabla_x F(x,y,z)| \leq C$;
4. $|\nabla_z F(x,y,z)| \leq C\,(1 + |z|)$;
5. $|\nabla_y F(x,y,z)| \leq C\,(1 + |z|)^{2\alpha}$.
6. $\lambda > 0$ and F is monotone in y in the following sense:

$$x \in H, y, y' \ \mathbf{R}, z \in \Xi^* \qquad < y - y', F(x,y,z) - F(x,y',z) > \leq -\lambda |y - y'|^2.$$

Applying Theorem 3.3, we obtain:

Proposition 4.4. *Let us suppose that Assumptions A2 and A3 hold. Then we have:*

(i) For any $x \in H$, there exists a solution (Y^x, Z^x) to the BSDE (14) such that Y^x is a continuous process bounded by K/λ, and $Z \in L^2_{\text{loc}}(\Omega; L^2(0,\infty;\Xi))$ with $\mathbb{E}\int_0^\infty e^{-2(\lambda+\epsilon)s}|Z_s|^2 ds < \infty$. The solution is unique in the class of processes (Y, Z) such that Y is continuous and bounded, and Z belongs to $L^2_{\text{loc}}(\Omega; L^2(0,\infty;\Xi))$.

(ii) *For all $T > 0$ and $p \geq 1$, the map $x \to (Y^x\big|_{[0,T]}, Z^x\big|_{[0,T]})$ is continuous from H to the space $L^p(\Omega; C(0,T;\mathbb{R})) \times L^p(\Omega; L^2(0,T;\Xi))$.*

Proof. Statement (i) is an immediate consequences of Theorem 3.3. Let us prove (ii). Denoting by $(Y^{n,x}, Z^{n,x})$ the unique solution of the following BSDE (with finite horizon):

$$Y_t^{n,x} = \int_t^n F(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x})d\sigma - \int_t^n Z_\sigma^{n,x} dW_\sigma, \tag{15}$$

then, from Theorem 3.3again, $|Y_t^{n,x}| \leq \frac{K}{\lambda}$ and the following convergence rate holds:

$$|Y_t^{n,x} - Y_t^x| \leq \frac{K}{\lambda} \exp\{-\lambda(n-t)\}.$$

Now, if $x'_m \to x$ as $m \to +\infty$ then

$$\begin{aligned} |Y_T^{x'_m} - Y_T^x| &\leq |Y_T^{x'_m} - Y_T^{n,x'_m}| + |Y_T^{n,x} - Y_T^x| + |Y_T^{n,x'_m} - Y_T^{n,x}| \\ &\leq 2\frac{K}{\lambda} \exp\{-\lambda(n-T)\} + |Y_T^{n,x'_m} - Y_T^{n,x}|. \end{aligned}$$

Moreover for fixed n, $Y_T^{n,x'_m} \to Y_T^{n,x}$ in $L^p(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ for all $p > 1$, by Proposition 4.2 in [2] Thus $Y_T^{x'_m} \to Y_T^x$ in $L^p(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

Now we can notice that $(Y^x\big|_{[0,T]}, Z^x\big|_{[0,T]})$ is the unique solution of the following BSDE (with finite horizon):

$$Y_t^x = Y_T^x + \int_t^T F(X_\sigma^x, Y_\sigma^x, Z_\sigma^x) - \int_t^T Z_\sigma^x dW_\sigma,$$

and the same holds for $(Y^{x'_m}\big|_{[0,T]}, Z^{x'_m}\big|_{[0,T]})$. By similar argument as in [2] we have

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^x - Y_t^{x'_m}|^p\right]^{1\wedge 1/p} + \mathbb{E}\left[\left(\int_0^T |Z_t^x - Z_t^{x'_m}|\right)^{p/2}\right]^{1\wedge 1/p} \\ &\leq C\,\mathbb{E}\left[\left|Y_T^x - Y_T^{x'_m}\right|^{p+1}\right]^{\frac{1}{p+1}} + \mathbb{E}\left[\left(\int_0^T \left|F(s, X_s^x, Y_s, Z_s) - F(s, X_s^{x'_m}, Y_s, Z_s)\right| ds\right)^{p+1}\right]^{\frac{1}{p+1}} \end{aligned}$$

and we can conclude that $(Y^{x'_m}\big|_{[0,T]}, Z^{x'_m}\big|_{[0,T]}) \to (Y^x\big|_{[0,T]}, Z^x\big|_{[0,T]})$ in $L^p(\Omega; C(0,T;\mathbb{R})) \times L^p(\Omega; L^2(0,T;\Xi))$. □

We need to study the regularity of Y^x. More precisely, we would like to show that Y_0^x belongs to $\mathcal{G}^1(H,\mathbb{R})$.

We are now in position to prove the main result of this section.

Theorem 4.5. *Under Assumption the map $x \to Y_0^x$ belongs to $\mathcal{G}^1(H,\mathbb{R})$. Moreover $|Y_0^x| + |\nabla_x Y_0^x| \leq c$, for a suitable constant c.*

Proof. Fix $n \geq 1$, let us consider the solution $(Y^{n,x}, Z^{n,x})$ of (15). Then, see [2], Proposition 4.2, the map $x \to (Y^{n,x}(\cdot), Z^{n,x}(\cdot))$ is Gâteaux differentiable from H to $L^p(\Omega,\ C(0,T;\mathbb{R})) \times$

$L^p(\Omega; L^2(0,T;\Xi^*))$, $\forall p \in (1,\infty)$. Denoting by $(\nabla_x Y^{n,x}h, \nabla_x Z^{n,x}h)$ the partial Gâteaux derivatives with respect to x in the direction $h \in H$, the processes $\{\nabla_x Y_t^{n,x}h, \nabla_x Z_t^{n,x}h, t \in [0,n]\}$ solves the equation, $\mathbb{P} - a.s.$,

$$\begin{aligned}
\nabla_x Y_t^{n,x}h &= \int_t^n \nabla_x F(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \nabla_x X_\sigma^{n,x} h \, d\sigma \\
&+ \int_t^n \nabla_y F(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \nabla_x Y_\sigma^{n,x} h \, d\sigma \\
&+ \int_t^n \nabla_z F(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \nabla_x Z_\sigma^{n,x} h \, d\sigma - \int_t^n \nabla_x Z_\sigma^{n,x} h \, dW_\sigma .
\end{aligned} \tag{16}$$

We note that we can write the generator of the previous equation as

$$\phi_\sigma^n(u,v) = \psi_\sigma^n + a_\sigma^n u + b_\sigma^n v$$

setting

$$\psi_\sigma^n = \nabla_x F(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \nabla_x X_\sigma^{n,x} h$$
$$a_\sigma^n = \nabla_y F(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \quad b_\sigma^n = \nabla_z F(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}).$$

By Assumption A3 and Lemma 4.3, we have that for all $x, h \in H$ the following holds $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$ and all $\sigma \in [0,n]$:

$$|\psi_\sigma^n| = \left| \nabla_x F(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \nabla_x X_\sigma^x h \right| \le C|h|,$$
$$a_\sigma^n = \nabla_y F(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \le -\lambda \le 0, \qquad |b_\sigma^n| = \left| \nabla_z F(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \right| \le C(1 + |Z_\sigma^{n,x}|).$$

Therefore $\int_0^\cdot Z_\sigma^{n,x} dW_\sigma$ is a BMO-martingale. Hence $\int_0^\cdot b_s dW_s$ is also a BMO-martingale and by Lemma 3.1, we obtain:

$$\sup_{t \in [0,n]} |\nabla_x Y_t^{n,x}| \le C|h|, \quad \mathbb{P} - \text{a.s.};$$

and applying Itô's formula to $e^{-2\lambda t} |\nabla_x Y_t^{n,x} h|^2$ and arguing as in the proof of Theorem 3.3, points (iii) and (iv), tanks to the (9), we get:

$$\mathbb{E} \int_0^\infty e^{-2\lambda t} (|\nabla_x Y_t^{n,x} h|^2 + |\nabla_x Z_t^{n,x} h|^2) dt \le C_1 |h|^2.$$

Fix $x, h \in H$, there exists a subsequence of $\{(\nabla_x Y^{n,x}h, \nabla_x Z^{n,x}h, \nabla_x Y_0^{n,x}h) : n \in \mathbb{N}\}$ which we still denote by itself, such that $(\nabla_x Y^{n,x}h, \nabla_x Z^{n,x}h)$ converges weakly to $(U^1(x,h), V^1(x,h))$ in $\mathcal{M}^{2,-2\lambda}(0,\infty; \mathbf{R} \times \Xi^*)$ and $\nabla_x Y_0^{n,x} h$ converges to $\xi(x,h) \in \mathbb{R}$.

Now we write the equation (16) as follows:

$$\begin{aligned}
\nabla_x Y_t^{n,x}h &= \nabla_x Y_0^{n,x} h - \int_0^t \nabla_x F(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \nabla_x X_\sigma^x h d\sigma \\
&- \int_0^t (\nabla_y F(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x})) \nabla_x Y_\sigma^{n,x} h d\sigma \\
&- \int_0^t \nabla_z F(X_\sigma^x, Y_\sigma^{n,x}, Z_\sigma^{n,x}) \nabla_x Z_\sigma^{n,x} h d\sigma + \int_0^t \nabla_x Z_\sigma^{n,x} h dW_\sigma
\end{aligned} \tag{17}$$

and define an other process $U_t^2(x,h)$ by

$$\begin{aligned}
U_t^2(x,h) &= \xi(x,h) - \int_0^t \nabla_x F(X_\sigma^x, Y_\sigma^x, Z_\sigma^x)\nabla_x X_\sigma^x\, h d\sigma \\
&\quad - \int_0^t (\nabla_y F(X_\sigma^x, Y_\sigma^x, Z_\sigma^x) U_\sigma^1(x,h) d\sigma \qquad (18)\\
&\quad - \int_0^t \nabla_z F(X_\sigma^x, Y_\sigma^x, Z_\sigma^x) V_\sigma^1(x,h) d\sigma + \int_0^t V_\sigma^1(x,h) dW_\sigma,
\end{aligned}$$

where (Y^x, Z^x) is the unique bounded solution to the backward equation (14), see Proposition 4.4. Passing to the limit in the equation (17) it is easy to show that $\nabla_x Y_t^{n,x} h$ converges to $U_t^2(x,h)$ weakly in $L^1(\Omega)$ for all $t > 0$.

Thus $U_t^2(x,h) = U_t^1(x,h)$, $\mathbb{P}$-a.s. for a.e. $t \in \mathbf{R}^+$ and $|U_t^2(x,h)| \le C|h|$.

Now consider the following equation on infinite horizon

$$\begin{aligned}
U(t,x,h) &= U(0,x,h) - \int_0^t \nabla_x F(X_\sigma^x, Y_\sigma^x, Z_\sigma^x)\nabla_x X_\sigma^x h d\sigma \\
&\quad - \int_0^t (\nabla_y F(X_\sigma^x, Y_\sigma^x, Z_\sigma^x)) U(t,x,h) d\sigma \qquad (19)\\
&\quad - \int_0^t \nabla_z F(X_\sigma^x, Y_\sigma^x, Z_\sigma^x) V(\sigma,x,h) d\sigma + \int_0^t V(\sigma,x,h) dW_\sigma.
\end{aligned}$$

We claim that this equation has a solution.

For each $n \in \mathbf{N}$ consider the finite horizon BSDE (with final condition equal to zero):

$$\begin{aligned}
U_n(t,x,h) &= \int_t^n \nabla_x F(X_\sigma^x, Y_\sigma^x, Z_\sigma^x)\nabla_x X_\sigma^x h d\sigma \\
&\quad + \int_t^n (\nabla_y F(X_\sigma^x, Y_\sigma^x, Z_\sigma^x)) U_n(t,x,h) d\sigma \\
&\quad + \int_t^n \nabla_z F(X_\sigma^x, Y_\sigma^x, Z_\sigma^x) V_n(\sigma,x,h) d\sigma - \int_t^n V_n(\sigma,x,h) dW_\sigma,
\end{aligned}$$

By the result in [2] we know that this equation has a unique solution $(U_n(\cdot,x,h), V_n(\cdot,x,h)) \in L^p(\Omega; C(0,n;\mathbb{R})) \times L^p(\Omega; L^2(0,n;\Xi^*))$. The generator of this equation can be rewrite as

$$\phi_t(u,v) = \psi_t + a_t u + b_t v$$

where $\psi_t = \nabla_x F(X_t^x, Y_t^x, Z_t^x)\nabla_x X_t^x$ and $|\psi_t| \le C|h|$, $a_t = \nabla_y F(X_\sigma^x, Y_\sigma^x, Z_\sigma^x) \le -\lambda$, $b_t = \nabla_z F(X_\sigma^x, Y_\sigma^x, Z_\sigma^x)$ and $|b_t| \le C(1 + |Z_t^x|)$. On the interval $[0,n]$ the process $\int_0^\cdot Z_s^x dWs$ is a BMO-martingale. Hence, from the Lemma 3.1 it follows that $\mathbb{P}$-a.s. $\forall n \in \mathbf{N}$, $\forall t \in [0,n]$ $|U_t^n| \le \frac{C}{\lambda}|h|$ and as in the proof of existence in the Theorem 3.3, we can conclude that

1. for each $t \ge 0$ $U^n(t,x,h)$ is a Cauchy sequence in $L^\infty(\Omega)$ which converges to a process U and $\mathbb{P}$-a.s., $\forall t \in [0,n]$

$$|U^n(t,x,h) - U(t,x,h)| \le \frac{C}{\lambda}|h| e^{-\lambda(n-t)};$$

2. $V^n(\cdot,x,h)$ is a Cauchy sequence in $L^2_{loc}(\Omega; L^2([0,\infty); \Xi^*)$;

3. The processes limit $(U(\cdot,x,h),V(\cdot,x,h)$ satisfy the BSDE (19).

Moreover still from Lemma 3.1 we get that the solution is unique.

Coming back to equation (18), we have that $(U^2(x,h),V^1(x,h))$ is solution in $\mathbb{R}^+$ of the equation (19).

In particular we notice that $U(0,x,h)=\xi(x,h)$ is the limit of $\nabla_x Y_0^{n,x}h$ (along the chosen subsequence). The uniqueness of the solution to (19) implies that in reality $U(0,x,h)=\lim_{n\to\infty}\nabla_x Y_0^{n,x}h$ along the original sequence.

Now let $x_m \to x$.

$$\begin{aligned}|U(0,x,h)-U(0,x_m,h)| &\le |U(0,x,h)-U^n(0,x,h)|+|U^n(0,x,h)-U^n(0,x_m,h)|+ \\ &+|U^n(0,x_m,h)-U(0,x_m,h)| \le \frac{2C}{\lambda}e^{-\lambda n}|h|+|U_n(0,x,h)-U_n(0,x_m,h)|,\end{aligned} \tag{20}$$

where we have used the (1). We now notice that $\nabla_x F$, $\nabla_y F$, $\nabla_z F$ are, by assumptions, continuous and $|\nabla_x F|\le C$, $|\nabla_y F|\le C(1+|Z|)^{2\alpha}$, $|\nabla_z F|\le C(1+|Z|)$. Moreover the following statements on continuous dependence on x hold:

maps $x\to X^x$, $x\to\nabla_x X^x h$ are continuous from $H\to L^p_{\mathcal{P}}(\Omega;C(0,T;H))$ (see [12] Proposition 3.3);

the map $x\to Y^x\big|_{[0,T]}$ is continuous from H to $L^p_{\mathcal{P}}(\Omega;C(0,T;\mathbb{R}))$ (see Proposition 4.4 here);

the map $x\to Z^x\big|_{[0,T]}$ is continuous from H to $L^p_{\mathcal{P}}(\Omega;L^2(0,T;\Xi))$ (see Proposition 4.4 here).

We can therefore apply to (20) the continuity result of [12] Proposition 4.3 to obtain in particular that $U_n(0,x'_m,h)\to U_n(0,x,h)$ for all fixed n as $m\to\infty$. And by (20) we can conclude that $U(0,x'_m,h)\to U(0,x,h)$ as $m\to\infty$.

Summarizing $U(0,x,h)=\lim_{n\to\infty}\nabla_x Y_0^{n,x}h$ exists, moreover it is clearly linear in h and verifies $|U(0,x,h)|\le C|h|$, finally it is continuous in x for every h fixed.

Finally, for $t>0$,

$$\begin{aligned}\lim_{t\searrow 0}\frac{1}{t}[Y_0^{x+th}-Y_0^x] &= \lim_{t\searrow 0}\frac{1}{t}\lim_{n\to+\infty}[Y_0^{n,x+th}-Y_0^{n,x}] = \lim_{t\searrow 0}\lim_{n\to+\infty}\int_0^1 \nabla_x Y_0^{n,x+\theta th}hd\theta \\ &= \lim_{t\searrow 0}\int_0^1 U(0,x+\theta th)hd\theta = U(0,x)h\end{aligned}$$

and the claim is proved. $\square$

5 Mild Solution of the elliptic PDE

Now we can proceed as in [13]. Let us consider the forward equation

$$X_s = e^{sA}x+\int_0^s e^{(s-r)A}b(X_r)dr+\int_0^s e^{(s-r)A}\sigma dW_r,\ s\ge 0. \tag{21}$$

Assuming that Assumption A2 holds, we define in the usual way the transition semigroup $(P_t)_{t\ge 0}$, associated to the process X:

$$P_t[\phi](x)=\mathbb{E}\,\phi(X_t^x),\qquad x\in H,$$

for every bounded measurable function $\phi : H \to \mathbf{R}$. Formally, the generator $\mathcal{L}$ of (P_t) is the operator

$$\mathcal{L}\phi(x) = \frac{1}{2}\mathrm{Trace}\left(\sigma\sigma^*\nabla^2\phi(x)\right) + \langle Ax + b(x), \nabla\phi(x)\rangle.$$

In this section we address solvability of the non linear stationary Kolmogorov equation:

$$\mathcal{L}v(x) + F(x, v(x), \nabla v(x)\,\sigma) = 0, \qquad x \in H, \tag{22}$$

when the coefficient F verifies Assumption A3. Note that, for $x \in H$, $\nabla v(x)$ belongs to H^*, so that $\nabla v(x)\,\sigma$ is in Ξ^*.

Definition 5.1. We say that a function $v : H \to \mathbf{R}$ is a mild solution of the non linear stationary Kolmogorov equation (22) if the following conditions hold:

(i) $v \in \mathcal{G}^1(H, \mathbf{R})$ and $\exists\, C > 0$ such that $|v(x)| \le C$, $|\nabla_x v(x)h| \le C\,|h|$, for all $x, h \in H$;

(ii) the following equality holds, for every $x \in H$ and $T \ge 0$:

$$v(x) = e^{-\lambda T}\ P_T[v](x) + \int_0^T e^{-\lambda t}\ P_t\Big[F\Big(\cdot, v(\cdot), \nabla v(\cdot)\,\sigma\Big) + \lambda v(\cdot)\Big](x)\ dt. \tag{23}$$

where λ is the monotonicity constant in Assumption A3.

Together with equation (21) we also consider the backward equation

$$Y_t - Y_T + \int_t^T Z_s dW_s = \int_t^T F(X_s, Y_s, Z_s)ds \qquad 0 \le t \le T < \infty \tag{24}$$

where $F : H \times \mathbf{R} \times \Xi^* \to \mathbf{R}$ is the same occurring in the nonlinear stationary Kolmogorov equation. Under the Assumptions A2, A3, Propositions 4.2-4.4 give a unique solution $\{X_t^x, Y_t^x, Z_t^x\}$, for $t \ge 0$, of the forward-backward system (21)-(24). We can now state the following

Theorem 5.2. *Assume that Assumption A2, Assumption A3 and hold then equation (22) has a unique mild solution given by the formula*

$$v(x) = Y_0^x.$$

where $\{X_t^x, Y_t^x, Z_t^x, t \ge 0\}$ is the solution of the forward-backward system (21)-(24). Moreover the following holds:

$$Y_t^x = v(X_t^x), \quad Z_t^x = \nabla v(X_t^x)\,\sigma.$$

Proof. Let us recall that for $s \ge 0$, Y_s^x is measurable with respect to $\mathcal{F}_{[0,s]}$ and $\mathcal{F}_s$; it follows that Y_0^x is deterministic (see also [7]). Moreover, as a byproduct of Proposition 4.5, the function v defined by the formula $v(x) = Y_0^x$ has the regularity properties stated in Definition 5.1. The proof that the equality (23) holds true for v is identical to the proof of Theorem 6.1 in [13].

6 Application to optimal control

We wish to apply the above results to perform the synthesis of the optimal control for a general nonlinear control system on an infinite time horizon. To be able to use non-smooth feedbacks

we settle the problem in the framework of weak control problems. Again we follow [13] with slight modifications. We report the argument for reader's convenience.

As above by H, Ξ we denote separable real Hilbert spaces and by U we denote a Banach space.

For fixed $x_0 \in H$ an *admissible control system* (a.c.s) is given by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, \{W_t, t \geq 0\}, u)$ where

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $(\mathcal{F}_t)_{t\geq 0}$ is a filtration on it satisfying the usual conditions.
- $\{W_t : t \geq 0\}$ is a Ξ-valued cylindrical Wiener process relatively to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and the probability $\mathbb{P}$.
- $u : \Omega \times [0, \infty[\to U$ is a predictable process (relatively to $(F_t)_{t\geq 0}$) that satisfies the constraint: $u_t \in \mathcal{U}$, $\mathbb{P}$-a.s. for a.e. $t \geq 0$, where $\mathcal{U}$ is a fixed closed subset of U.

To each a.c.s. we associate the mild solution $X \in L^r_{\mathcal{P}}(\Omega; C(0, T; H))$ (for arbitrary $T > 0$ and arbitrary $r \geq 1$) of the state equation:

$$\begin{cases} dX_\tau = (AX_\tau + b(X_\tau) + \sigma r(X_\tau, u_\tau))\ d\tau + \sigma\ dW_\tau, & \tau \geq 0, \\ X_0 = x \in H, \end{cases} \tag{25}$$

and the cost:

$$J(x, u) = \mathbb{E} \int_0^{+\infty} e^{-\lambda t} g(X_t, u_t)\ dt, \tag{26}$$

where $g : H \times U \to \mathbf{R}$. Our purpose is to minimize the functional J over all a.c.s. Notice the occurrence of the operator σ in the control term: this special structure of the state equation is imposed by our techniques.

We work under the following assumptions.

Assumption A4. 1. The process W is a Wiener process in Ξ, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $(\mathcal{F}_t)$ satisfying the usual conditions.

2. A, b verify Assumption A2.
3. σ satisfies Assumption A2 (iii) with $\gamma = 0$;
4. The set $\mathcal{U}$ is a nonempty closed subset of U .
5. The functions $r : H \times U \to \Xi$, $g : H \times U \to \mathbf{R}$ are Borel measurable and for all $x \in H$, $r(x, \cdot)$ and $g(x, \cdot)$ are continuous functions from U to Ξ and from U to $\mathbf{R}$, respectively.
6. There exists a constant $C \geq 0$ such that for every $x, x' \in H$, $u \in K$ it holds that

$$|r(x, u) - r(x', u)| \leq C(1 + |u|)|x - x'|,$$

$$|r(x, u)| \leq C(1 + |u|), \tag{27}$$

$$0 \leq g(x, u) \leq C(1 + |u|^2), \tag{28}$$

7. There exist $R > 0$ and $c > 0$ such that for every $x \in H$ $u \in U$ satisfying $|u| \geq R$,

$$g(x, u) \geq c|u|^2. \tag{29}$$

We will say that an $(\mathcal{F}_t)$-adapted stochastic process $\{u_t, t \geq 0\}$ with values in U is an admissible control if it satisfies

$$\mathbb{E}\int_0^\infty e^{-\lambda t}|u_t|^2 dt < \infty. \tag{30}$$

This square summability requirement is justified by (29): a control process which is not square summable would have infinite cost.

Now we state that for every admissible control the solution to (25) exists.

Proposition 6.1. *Let u be an admissible control. Then there exists a unique, continuous, $(\mathcal{F}_t)$-adapted process X satisfying $\mathbb{E}\sup_{t\in[0,T]}|X_t|^2 < \infty$, and $\mathbb{P}$-a.s., $t \in [0,T]$*

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}b(X_s)ds + \int_0^t e^{(t-s)A}\sigma dW_s + \int_0^t e^{(t-s)A}\sigma r(X_s, u_s)ds.$$

Proof. The proof is an immediate extension to the infinite dimensional case of the Proposition 2.3 in [11]. □

By the previous Proposition and the arbitrariness of T in its statement, the solution is defined for every $t \geq 0$. We define in a classical way the Hamiltonian function relative to the above problem: for all $x \in H$, $z \in \Xi^*$,

$$\begin{aligned} F(x,y,z) &= \inf\{g(x,u) + zr(x,u) : u \in \mathcal{U}\} - \lambda y \\ \Gamma(x,y,z) &= \{u \in U : g(x,u) + zr(x,u) - \lambda y = F(x,y,z)\}. \end{aligned} \tag{31}$$

The proof of the following Lemma can be found in [11] Lemma 3.1.

Lemma 6.2. *The map F is a Borel measurable function from $H \times \Xi^*$ to $\mathbf{R}$. There exists a constant $C > 0$ such that*

$$-C(1+|z|^2) - \lambda y \leq F(x,y,z) \leq g(x,u) + C|z|(1+|u|) - \lambda y \quad \forall u \in \mathcal{U}. \tag{32}$$

We require moreover that

Assumption A5. F satisfies assumption A3 2-3-4.

We notice that the cost functional is well defined and $J(x,u) < \infty$ for all $x \in H$ and all a.c.s.

By Theorem 5.2, the stationary Hamilton-Jacobi-Bellman equation relative to the above stated problem, namely:

$$\mathcal{L}v(x) + F(x, v(x), \nabla v(x)\sigma) = 0, \qquad x \in H, \tag{33}$$

admits a unique mild solution, in the sense of Definition 5.1.

6.0.1 The fundamental relation

Proposition 6.3. *Let v be the solution of (33). For every admissible control u and for the corresponding trajectory X starting at x we have*

$$\begin{aligned} &J(x,u) = v(x)+ \\ &\mathbb{E}\int_0^\infty e^{-\lambda t}\Big(- F(X_t, \nabla v(X_t)\sigma) - \lambda v(X_t) + \nabla_x v(X_t)\sigma r(X_t,u_t) + g(X_t,u_t)\Big)\, dt. \end{aligned}$$

Proof. We introduce the sequence of stopping times

$$\tau_n = \inf\{t \in [0,T] : \int_0^t |u_s|^2 ds \geq n\},$$

with the convention that $\tau_n = T$ if the indicated set is empty. By (30), for $\mathbb{P}$-almost every $\omega \in \Omega$, there exists an integer $N(\omega)$ depending on ω such that

$$n \geq N(\omega) \Longrightarrow \tau_n(\omega) = T. \tag{34}$$

Let us fix $u_0 \in K$, and for every n, let us define

$$u_t^n = u_t 1_{t \leq \tau_n} + u_0 1_{t > \tau_n}$$

and consider the equation

$$\begin{cases} dX_t^n = b(X_t^n)dt + \sigma[dW_t + r(X_t^n, u_t^n)dt], \quad 0 \leq t \leq T \\ X_0^n = x. \end{cases} \tag{35}$$

Let us define

$$W_t^n := W_t + \int_0^t r(X_s^n, u_s^n)ds \quad 0 \leq t \leq T.$$

From the definition of τ_n and from (27), it follows that

$$\int_0^T |r(X_s^n, u_s^n)|^2 ds \leq C \int_0^T (1 + |u_s^n|)^2 ds \leq C \int_0^{\tau_n} (1 + |u_s|)^2 ds + C \leq C + Cn. \tag{36}$$

Therefore defining

$$\rho_n = \exp\left(\int_0^T -r(X_s^n, u_s^n)dW_s - \frac{1}{2} \int_0^T |r(X_s^n, u_s^n)|^2 ds \right)$$

the Novikov condition implies that $\mathbb{E}\rho_n = 1$. Setting $d\mathbb{P}_T^n = \rho_n d\mathbb{P}_{|\mathcal{F}_T}$, by the Girsanov theorem W^n is a Wiener process under $\mathbb{P}_T^n$. Relatively to W^n the equation (35) can be written:

$$\begin{cases} dX_t^n = b(X_t^n)dt + \sigma dW_t^n, \quad 0 \leq t \leq T \\ X_0^n = x. \end{cases} \tag{37}$$

Consider the following finite horizon Markovian forward-backward system (with respect to probability $\mathbb{P}_T^n$ and to the filtration generated by $\{W_\tau^n : \tau \in [0,T]\}$).

$$\begin{cases} X_\tau^n(x) = e^{\tau A}x + \displaystyle\int_0^\tau e^{(\tau-s)A} b(X_s^n(x))\, ds + \int_0^\tau e^{(\tau-s)A}\sigma\, dW_s^n, \quad \tau \geq 0, \\ Y_\tau^n(x) - v(X_T^n(x)) + \displaystyle\int_t^T Z_s^n(x) dW_s^n = \int_t^T F(X_s^n(x), Y_s^n(x), Z_s^n(x))ds,\ 0 \leq \tau \leq T, \end{cases} \tag{38}$$

and let $(X^n(x), Y^n(x), Z^n(x))$ be its unique solution with the three processes predictable relatively to the filtration generated by $\{W_\tau^n : \tau \in [0,T]\}$ and: $\mathbb{E}_T^n \sup_{t\in[0,T]} |X_t^n(x)|^2 < +\infty$, $Y^n(x)$ bounded and continuous, $\mathbb{E}_T^n \int_0^T |Z_t^n(x)|^2 dt < +\infty$. Moreover, Theorem 5.2 and uniqueness of the solution of system (38), yields that

$$Y_t^n(x) = v(X_t^n(x)), \qquad Z_t^n(x) = \nabla v(X_t^n(x)) G(X_t^n(x)). \tag{39}$$

Applying the Itô formula to $e^{-\lambda t}Y_t^n(x)$, and restoring the original noise W we get

$$\begin{aligned} e^{-\lambda\tau_n}Y_{\tau_n}^n(x) &= e^{-\lambda T}Y_T^n(x) + \int_{\tau_n}^T \lambda e^{-\lambda t}Y_s^n(x)ds - \int_{\tau_n}^T e^{-\lambda s}Z_s^n(x)\, dW_s \\ &+ \int_{\tau_n}^T e^{-\lambda s}\left[F(X_s^n(x), Y_s^n(x), Z_s^n(x)) - Z_s^n(x)r(X_s^n, u_s^n)\right]\, ds. \end{aligned} \tag{40}$$

We note that for every $p \in [1, \infty)$ we have

$$\rho_n^{-p} = \exp\left(p\int_0^T r(X_s^n, u_s^n)dW_s^n - \frac{p^2}{2}\int_0^T |r(X_s^n, u_s^n)|^2 ds\right) \cdot \exp\left(\frac{p^2 - p}{2}\int_0^T |r(X_s^n, u_s^n)|^2 ds\right). \tag{41}$$

By (36) the second exponential is bounded by a constant depending on n and p, while the first one has $\mathbb{P}^n$-expectation, equal to 1. So we conclude that $\mathbb{E}^n\rho_n^{-p} < \infty$. It follows that

$$\mathbb{E}\left(\int_0^T e^{-2\lambda t}|Z_t^n(x)|^2 dt\right)^{1/2} \leq \mathbb{E}^n\left(\int_0^T \rho_n^{-2}|Z_t^n(x)|^2 dt\right)^{1/2} \leq$$
$$\leq (\mathbb{E}^n\rho_n^{-2})^{1/2}\mathbb{E}^n\left(\int_0^T |Z_t^n(x)|^2 dt\right)^{1/2} < \infty$$

We conclude that the stochastic integral in (40) has zero expectation. Using the identification in (39) and taking expectation with respect to $\mathbb{P}$, we obtain

$$\begin{aligned} \mathbb{E}e^{-\lambda\tau_n}Y_{\tau_n}^n &= e^{-\lambda T}\mathbb{E}[v(X_T^n(x))] + \mathbb{E}\int_{\tau_n}^T \lambda e^{-\lambda t}Y_s^n(x)ds + \\ &+ \mathbb{E}\int_{\tau_n}^T e^{-\lambda s}\left[F(X_s^n(x), Y_s^n(x), Z_s^n(x)) - Z_s^n(x)r(X_s^n(x), u_s^n)\right]\, ds \leq \\ &\leq e^{-\lambda T}\mathbb{E}[v(X_T^n(x)] + \mathbb{E}\int_{\tau_n}^T \lambda e^{-\lambda s}Y_s^n(x)ds + \mathbb{E}\int_{\tau_n}^T e^{-\lambda s}g(X_s^n(x), u_s^n)ds. \end{aligned} \tag{42}$$

Now we let $n \to \infty$. By Proposition 4.4,

$$\sup_{t\geq 0}|Y_t^n| = \sup_{t\geq 0}|v(X_t^n)| \leq \frac{K}{\lambda}; \tag{43}$$

in particular

$$\mathbb{E}\int_{\tau_n}^T \lambda e^{-\lambda s}Y_s^n(x)ds \leq \mathbb{E}\int_{\tau_n}^T \lambda e^{-\lambda s}\frac{K}{\lambda}ds \leq \mathbb{E}K(T - \tau_n)$$

and the right-hand side tends to 0 by (34). By the definition of u^n and (28),

$$\mathbb{E}\int_{\tau_n}^T g(X_s^n, u_s^n)ds = \mathbb{E}\int_0^T 1_{s>\tau_n}g(X_s^n, u_0)ds \leq$$
$$\leq C\mathbb{E}\int_0^T 1_{s>\tau_n}(1 + |u_0|^2)ds \leq C\mathbb{E}(T - \tau_n) \tag{44}$$

and the right-hand side tends to 0 again by (34). Next we note that, again by (34), for $n \geq N(\omega)$ we have $\tau_n(\omega) = T$ and $v(X_T^n) = v(X_{\tau_n}^n) = v(X_{\tau_n}) = v(X_T)$. We deduce, thanks to (43), that $\mathbb{E}v(X_T^n) \to \mathbb{E}v(X_T)$, and from (42) we conclude that

$$\limsup_{n\to\infty} \mathbb{E}e^{-\lambda\tau_n} Y_{\tau_n}^n \leq e^{-\lambda T}\mathbb{E}v(X_T).$$

On the other hand, for $n \geq N(\omega)$ we have $\tau_n(\omega) = T$ and $e^{-\lambda\tau_n}Y_{\tau_n}^n = e^{-\lambda T}Y_T^n = e^{-\lambda T}v(X_T^n) = e^{-\lambda T}v(X_T)$. Since Y^n is bounded, by the Fatou lemma, $\mathbb{E}e^{-\lambda T}v(X_T) \leq \liminf_{n\to\infty} \mathbb{E}e^{-\lambda\tau_n}Y_{\tau_n}^n$. We have thus proved that

$$\lim_{n\to\infty} \mathbb{E}e^{-\lambda\tau_n} Y_{\tau_n}^n = e^{-\lambda T}\mathbb{E}v(X_T). \tag{45}$$

Now we return to backward equation in the system (38) and write

$$\begin{aligned} &e^{-\lambda\tau_n}Y_{\tau_n}^n = Y_0^n + \\ &+ \int_0^{\tau_n} -e^{-\lambda t}F(X_t^n, Y_t^n, Z_t^n)dt + \int_0^{\tau_n} -\lambda e^{-\lambda t}Y_t^n dt + \int_0^{\tau_n} e^{-\lambda t}Z_t^n dW_t + \int_0^{\tau_n} e^{-\lambda t}Z_t^n r(X_t^n, u_t^n)dt \end{aligned}$$

Arguing as before, we conclude that the stochastic integral has zero $\mathbb{P}$-expectation. Moreover, we have $Y_0^n = v(x)$, and, for $t \leq \tau_n$, we also have $u_t^n = u_t$, $X_t^n = X_t$, $Y_t^n = v(X_t^n) = v(X_t)$ and $Z_t^n = \nabla_x v(X_t)$. Thus, we obtain

$$\begin{aligned} &\mathbb{E}[e^{-\lambda\tau_n}Y_{\tau_n}^n] = v(x) + \\ &\qquad + \mathbb{E}\int_0^{\tau_n} e^{-\lambda t}\Big(- F(X_t, v(X_t), \nabla_x v(X_t)\sigma) - \lambda v(X_t) + \nabla_x v(X_t)\sigma r(X_t, u_t)\Big)dt \end{aligned} \tag{46}$$

and

$$\begin{aligned} &\mathbb{E}\int_0^{\tau_n} e^{-\lambda t}g(X_t, u_t)dt + \mathbb{E}[e^{-\lambda\tau_n}Y_{\tau_n}^n] = v(x) + \\ &+ \mathbb{E}\int_0^{\tau_n} e^{-\lambda t}\Big(- F(X_t, v(X_t), \nabla_x v(X_t)\sigma) - \lambda v(X_t) + \nabla_x v(X_t)\sigma r(X_t, u_t) + g(X_t, u_t)\Big)dt. \end{aligned} \tag{47}$$

Noting that $-F(x, y, z) - \lambda y + zr(x, u) + g(x, u) \geq 0$ and recalling that $g(x, u) \geq 0$ by (45) and the monotone convergence theorem, we obtain for $n \to \infty$,

$$\begin{aligned} &\mathbb{E}\int_0^{T} e^{-\lambda t}g(X_t, u_t)dt + e^{-\lambda T}\mathbb{E}v(X_T) = v(x) + \\ &\qquad + \mathbb{E}\int_0^{T} e^{-\lambda t}\Big(- F(X_t, \nabla_x v(X_t)\sigma) - \lambda v(X_t) + \nabla_x v(X_t)\sigma r(X_t, u_t) + g(X_t, u_t)\Big)dt. \end{aligned} \tag{48}$$

Recalling that v is bounded, letting $T \to \infty$, we conclude

$$\begin{aligned} &J(x, u) = v(x) + \\ &\mathbb{E}\int_0^{\infty} e^{-\lambda t}\left[-F(X_t, v(X_t), \nabla v(X_t)\sigma) - \lambda v(X_t) + \nabla_x v(X_t)\sigma r(X_t, u_t) + g(X_t, u_t)\right]dt. \end{aligned}$$

The above equality is known as the *fundamental relation* and immediately implies that $v(x) \leq J(x, u)$ and that the equality holds if and only if the following feedback law holds $\mathbb{P}$-a.s. for almost every $t \geq 0$:

$$F(X_t, v(X_t), \nabla_x v(X_t)\sigma) = \nabla_x v(X_t)\sigma + g(X_t, u_t) - \lambda v(X_t)$$

where X is the trajectory starting at x and corresponding to control u. □

6.0.2 Existence of optimal controls: the closed loop equation.

Next we address the problem of finding a weak solution to the so-called closed loop equation. We have to require the following

Assumption A6. $\Gamma(x,y,z)$, defined in 31, is non empty for all $x \in H$ and $z \in \Xi^*$.

By simple calculation (see [11] Lemma 3.1), we can prove that this infimum is attained in a ball of radius $C(1+|z|)$, that is,

$$F(x,y,z) = \min_{u \in \mathcal{U}, |u| \leq C(1+|z|)} [g(x,u) + zr(x,u)] - \lambda y, \quad x \in H,\, y \in \mathbf{R},\, z \in \Xi^*,$$

and

$$F(x,y,z) < g(x,u) + zr(x,u) - \lambda y \quad \text{ if } |u| > C(1+|z|). \tag{49}$$

Moreover, by the Filippov Theorem (see, e.g., [1, Thm. 8.2.10, p. 316]) there exists a measurable selection of Γ, a Borel measurable function $\gamma : H \times \Xi^* \to \mathcal{U}$ such that

$$F(x,y,z) = g(x,\gamma(x,z)) + zr(x,\gamma(x,z)) - \lambda y, \quad x \in H,\, y \in \mathbf{R},\, z \in \Xi^*. \tag{50}$$

By (49), we have

$$|\gamma(x,z)| \leq C(1+|z|). \tag{51}$$

We define

$$\underline{u}(x) = \gamma(x, \nabla_x v(X_t)\sigma) \quad \mathbb{P}\text{-}a.s. \text{ for } a.e\ t \geq 0.$$

The closed loop equation is

$$\begin{cases} dX_t = AX_t dt + b(X_t)dt + \sigma[dWt + r(X_t, \underline{u}(X_t))dt] & t \geq 0 \\ X_0 = x \end{cases} \tag{52}$$

By a weak solution we mean a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions, a Wiener process W in Ξ with respect to $\mathbb{P}$ and $(\mathcal{F}_t)$, and a continuous $(\mathcal{F}_t)$-adapted process $X$with values in H satisfying, $\mathbb{P}$-a.s.,

$$\int_0^\infty e^{-\lambda t} |\underline{u}(X_t)|^2 dt < \infty$$

and such that (52) holds. We note that by (27) it also follows that

$$\int_0 |r(X_t, \underline{u}(X_t))|^2 dt < \infty, \quad \mathbb{P} - a.s.,$$

so that (52) makes sense.

Proposition 6.4. *Assume that b, σ, g satisfy Assumption A4, F verifies Assumption A5 and Assumption A6 holds. Then there exists a weak solution of the closed loop equation, satisfying in addition*

$$\mathbb{E} \int_0^\infty e^{-\lambda t} |\underline{u}(X_t)|^2 dt < \infty. \tag{53}$$

Proof. We start by constructing a canonical version of a cylindrical Wiener process in Ξ. An explicit construction is needed to clarify the application of an infinite-dimensional version of the Girsanov theorem that we use below. We choose a larger Hilbert space $\Xi^{'} \supset \Xi$ in such a way that Ξ is continuously and densely embedded in $\Xi^{'}$ with Hilbert-Schmidt inclusion operator $\mathcal{J}$. By Ω we denote the space $C([0,\infty[,\Xi^{'})$ of continuous functions $\omega : [0,\infty[\to \Xi^{'}$ endowed with the usual locally convex topology that makes Ω a Polish space, and by $\mathcal{B}$ its Borel σ-field. Since $\mathcal{J}\mathcal{J}^*$ has finite trace on $\Xi^{'}$, it is well known that there exists a probability $\mathbb{P}$ on $\mathcal{B}$ such that the canonical processes $W_t^{'}(\omega) := \omega(t)$, $t \geq 0$, is a Wiener process with continuous paths in $\Xi^{'}$ satisfying $\mathbb{E}[\langle W_t^{'}, \xi^{'}\rangle_{\Xi^{'}} \langle W_s^{'}, \eta^{'}\rangle_{\Xi^{'}}] = \langle \mathcal{J}\mathcal{J}^* \xi^{'}, \eta^{'}\rangle_{\Xi^{'}} (t \wedge s)$ for all $\xi^{'}, \eta^{'} \in \Xi^{'}$, $t, s \geq 0$. This is called a $\mathcal{J}\mathcal{J}^*$-Wiener processes in $\Xi^{'}$ in [8], to which we refer the reader for preliminary material on Wiener processes on Hilbert spaces. Let us denote by $\mathcal{G}$ the $\mathbb{P}$-completion of $\mathcal{B}$ and by $\mathcal{N}$ the family of sets $A \in \mathcal{G}$ with $\mathbb{P}(A) = 0$. Let $\mathcal{B}_t = \sigma\{W_s^{'} : s \in [0,t]\}$ and $\mathcal{F}_t = \sigma(\mathcal{B}_t, \mathcal{N})$, $t \geq 0$, where as usual $\sigma(\cdot)$ denotes the σ-algebra in Ω generated by the indicated collection of sets or random variables. Thus $(\mathcal{F}_t)_{t\geq 0}$ is the Brownian filtration of $W^{'}$.

The Ξ-valued cylindrical Wiener process $\{W_t^{\xi} : t \geq 0, \xi \in \Xi\}$ can now be defined as follows. For ξ in the image of $\mathcal{J}^*\mathcal{J}$ we take η such that $\xi = \mathcal{J}^*\mathcal{J}\eta$ and define $W_s^{\xi} = \langle W_s^{'}, \mathcal{J}\eta\rangle_{\Xi^{'}}$. Then we notice that $\mathbb{E}|W_t^{\xi}|^2 = t|\mathcal{J}\eta|^2_{\Xi^{'}} = t|\xi|^2_{\Xi}$, which shows that the mapping $\xi \to W_s^{\xi}$, defined for $\xi \in \mathcal{J}^*\mathcal{J}(\Xi) \subset \Xi$ with values in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, is an isometry for the norms of Ξ and $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Consequently, noting that $\mathcal{J}^*\mathcal{J}(\Xi)$ is dense in Ξ, it extends to an isometry $\xi \to L^2(\omega, \mathcal{F}, \mathbb{P})$, still denoted $\xi \to W_s^{\xi}$. An appropriate modification of $\{W_t^{\xi} : t \geq 0, \xi \in \Xi\}$ gives the required cylindrical Wiener process. We note that the Brownian filtration of W coincides with $(\mathcal{F}_t)_{t\geq 0}$.

Now let $X \in L^p_{\text{loc}}(\Omega, C(0,+\infty; H))$ be the mild solution of

$$\begin{cases} dX_\tau = AX_\tau \, d\tau + b(X_\tau)\, d\tau + \sigma \, dW_\tau \\ X_0 = x \end{cases} \tag{54}$$

If together with previous forward equation we also consider the backward equation

$$Y_t - Y_T + \int_t^T Z_s dW_s = \int_t^T F(X_s, Y_s, Z_s) ds \qquad 0 \leq t \leq T < \infty \tag{55}$$

we know that there exists a unique solution $\{X_t^x, Y_t^x, Z_t^x, t \geq 0\}$ forward-backward system (54)-(55) and by Proposition 5.2,

$$v(x) = Y_0^x.$$

is the solution of the of the non linear stationary Kolmogorov equation:

$$\mathcal{L}v(x) + F(x, v(x), \nabla v(x)\, \sigma) = 0, \qquad x \in H. \tag{56}$$

Moreover the following holds:

$$Y_\tau(x) = v(X_\tau(x)), \qquad Z_\tau(x) = \nabla v(X_\tau(x))\sigma \tag{57}$$

We have

$$\mathbb{E} \int_0^\infty e^{-(\lambda+\epsilon)t} |Z_t|^2 dt < \infty. \tag{58}$$

and hence

$$\mathbb{E} \int_0^T |Z_t|^2 dt < \infty. \tag{59}$$

By (27) we have

$$|r(X_t, \underline{u}(X_t))| \leq C(1 + |\underline{u}(X_t)|), \tag{60}$$

and by (51),

$$|\underline{u}(X_t)| = |\gamma(X_t, \nabla v(X_t(x))\sigma)| \leq C(1 + |\nabla v(X_t(x))\sigma|) = C(1 + |Z_t|). \tag{61}$$

Let us define $\forall T > 0$

$$M_T = \exp\left(\int_0^T \langle r(X_s, \underline{u}(X_s), dW_s\rangle_\Xi - \frac{1}{2}\int_0^T |r(X_s, \underline{u}(X_s)|_\Xi^2 \, ds\right). \tag{62}$$

Now, arguing exactly as in the proof of Proposition 5.2 in [11], we can prove that $\mathbb{E}M_T = 1$, and M is a $\mathbb{P}$-martingale. Hence there exists a probability $\widehat{\mathbb{P}}_T$ on $\mathcal{F}_T$ admitting M_T as a density with respect to $\mathbb{P}$, and by the Girsanov Theorem we can conclude that $\{\widehat{W}_t, t \in [0,T]\}$ is a Wiener process with respect to $\mathbb{P}$ and $(\mathcal{F}_t)$. Since Ξ' is a Polish space and $\widehat{\mathbb{P}}_{T+h}$ coincide with $\widehat{\mathbb{P}}_T$ on $\mathcal{B}_T$, $T, h \geq 0$, by known results (see [22], Chapter VIII, §1, Proposition (1.13)) there exists a probability $\widehat{\mathbb{P}}$ on $\mathcal{B}$ such that the restriction on $\mathcal{B}_T$ of $\widehat{\mathbb{P}}_T$ and that of $\widehat{\mathbb{P}}$ coincide, $T \geq 0$. Let $\widehat{\mathcal{G}}$ be the $\widehat{\mathbb{P}}$-completion of $\mathcal{B}$ and $\widehat{\mathcal{F}}_T$ be the $\widehat{\mathbb{P}}$-completion of $\mathcal{B}_T$. Moreover, since for all $T > 0$, $\{\widehat{W}_t : t \in [0,T]\}$ is a Ξ-valued cylindrical Wiener process under $\widehat{\mathbb{P}}_T$ and the restriction of $\widehat{\mathbb{P}}_T$ and of $\widehat{\mathbb{P}}$ coincide on $\mathcal{B}_T$ modifying $\{\widehat{W}_t : t \geq 0\}$ in a suitable way on a $\widehat{\mathbb{P}}$-null probability set we can conclude that $(\Omega, \widehat{\mathcal{G}}, \{\widehat{\mathcal{F}}_t, t \geq 0\}, \widehat{\mathbb{P}}, \{\widehat{W}_t, t \geq 0\}, \gamma(X, \nabla v(X)\sigma(X)))$ is an admissible control system. The above construction immediately ensures that, if we choose such an admissible control system, then (52) is satisfied. Indeed if we rewrite (54) in terms of $\{\widehat{W}_t : t \geq 0\}$ we get

$$\begin{cases} dX_\tau = AX_\tau \, d\tau + b(X_\tau) \, d\tau + \sigma \left[r(X_\tau, \underline{u}(X_\tau))d\tau + d\widehat{W}_\tau\right] \\ X_0 = x. \end{cases}$$

It remains to prove (53). We define stopping times

$$\sigma_n = \inf\left\{t \geq 0 : \int_0^t e^{-\lambda t}|Z_s|^2 ds \geq n\right\},$$

with the convention that $\sigma_n = \infty$ if the indicated set is empty. By (58) for $\mathbb{P}$-a.s. $\omega \in \Omega$ there exists an integer $N(\omega)$ depending on ω such that $\sigma_n(\omega) = \infty$ for $n \geq N(\omega)$. Applying the Ito formula to $e^{-\lambda t}Y_t$, with respect to W , we obtain

$$\begin{aligned} e^{-\lambda\sigma_n}Y_{\sigma_n} &= Y_0 - \int_0^{\sigma_n} e^{-\lambda s} Z_s \, dW_s + \\ &+ \int_0^{\sigma_n} e^{-\lambda s}\left[-F(X_s, Y_s, Z_s) - \lambda Y_s(x) ds + Z_s r(X_s, \underline{u}(X_s))\right] \, ds. \end{aligned}$$

from which we deduce that

$$\begin{aligned} &\mathbb{E}e^{-\lambda\sigma_n}Y_{\sigma_n} + \mathbb{E}\int_0^{\sigma_n} e^{-\lambda s} g(X_s, \underline{u}(X_s)) ds = Y_0 + \\ &\quad + \mathbb{E}\int_0^{\sigma_n} e^{-\lambda s}\left[-F(X_s, Y_s, Z_s) - \lambda Y_s ds + Z_s r(X_s, \underline{u}(X_s)) + g(X_s, \underline{u}(X_s))\right] ds = Y_0. \end{aligned}$$

with the last equality coming from the definition of $\underline{u}$. Recalling that Y is bounded, it follows that

$$E\int_0^{\sigma_n} e^{-\lambda s} g(X_s, \underline{u}(X_s)) ds \leq C$$

for some constant C independent of n. By (29) and by sending n to infinity, we finally prove (53).

□

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